Instrumentation and Information Processing

Chapter 1: Signal and noise

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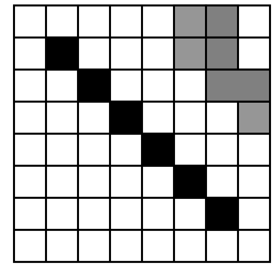
Contents

Providing thinking tools for design and evaluation of measurement systems

Measurement of Pattern

１） Spatial pattern

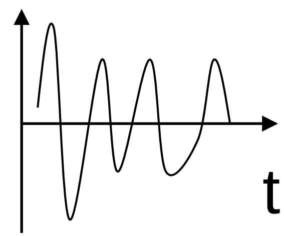


Image

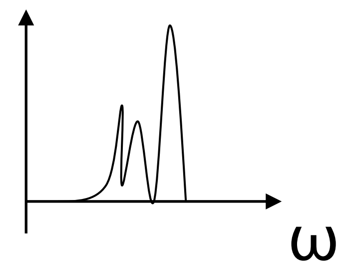
Picture -- 2D

Movie --- 3D

２） Temporal pattern



Time



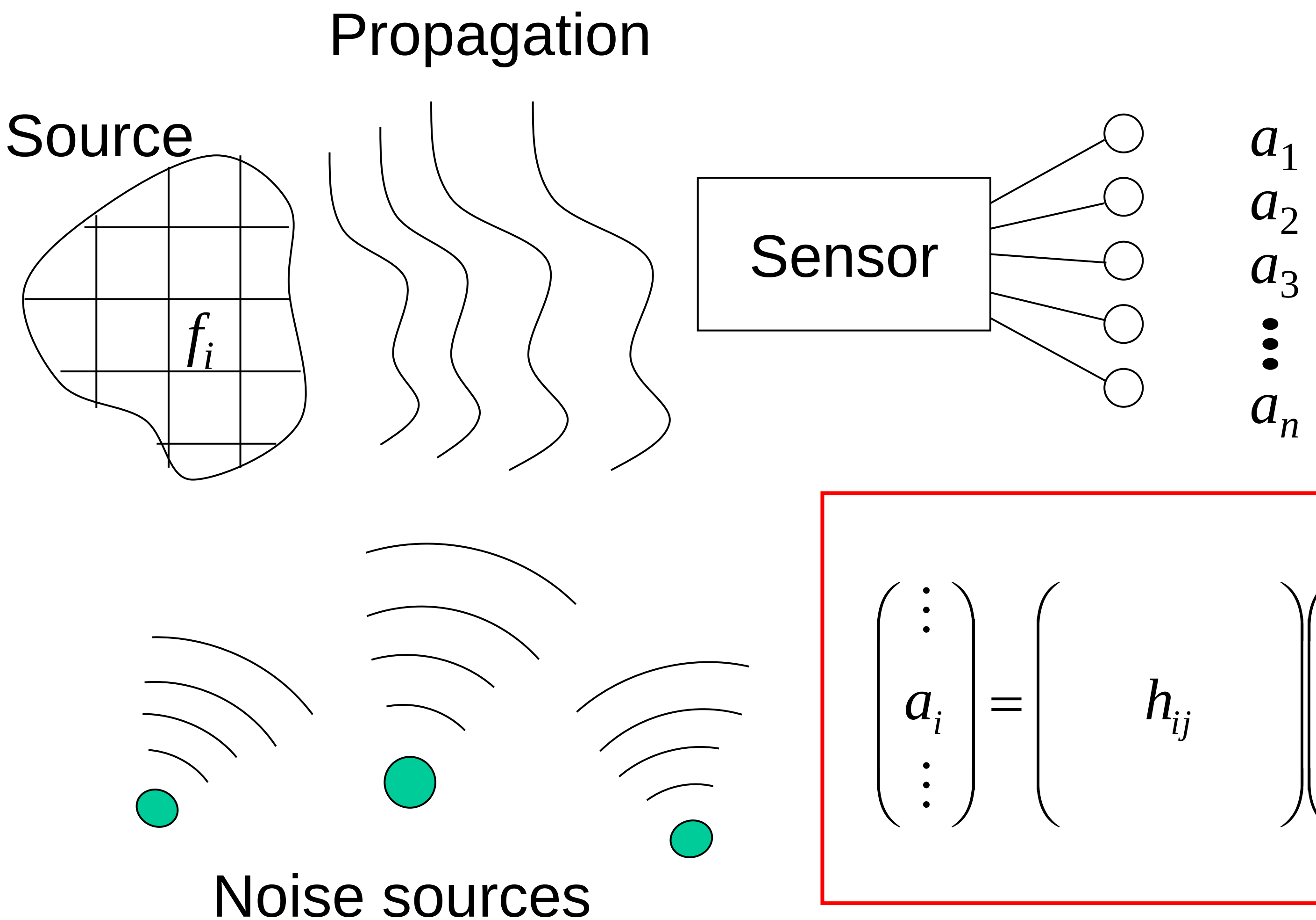
Frequency

３） Pattern of physical parameters

Optical spectrum, force vector, etc.

４） Composite of the above

Pattern measurement in linear system



Noise

$$\begin{pmatrix} \vdots \\ a_i \\ \vdots \end{pmatrix} = \begin{pmatrix} & & \\ & h_{ij} & \\ & & \end{pmatrix} \begin{pmatrix} \vdots \\ f_j \\ \vdots \end{pmatrix} + \begin{pmatrix} \vdots \\ w_i \\ \vdots \end{pmatrix}$$

Obtain $\boldsymbol{f}$ from $\boldsymbol{a}$

Classification of pattern measurement

Physical parameter	Phenomenon used	Information needed
• Temperature • Electromagnetic wave • Electric/magnetic field • Acoustic field • Force/pressure/stress • Displacement, vibration (solid, liquid, gas) • Velocity, acceleration, angular velocity, position • Elasticity, viscosity • Time, frequency	• Wave reflection, transmission, absorption • X-ray absorption • resonance • Nuclear magnetic resonance • Ohm's law • Photoelectric effect • interference • Moire • Black body radiation • Thermoelectric effect • Hook's law • Piezoelectric effect • Tunneling Effect ⋮	• Cloud distribution • Vegetation of forest • Location and spectrum of stars • Surface temperature distribution • Driving condition of car • State of living tissue • Thinking brain state • Object shape • 3D model of the environment • Driver's fatigue • Robot position • Human behavior ⋮

Measurement vs. communication

[Measurement]   Obtain quantities of (mainly) natural object

[Receiving signal in communication]

Obtain the signal that was intentionally sent from the sender

Phone, Wi-Fi, Bluetooth, ITS, IC tag, …

[Mixture of measurement & communication]

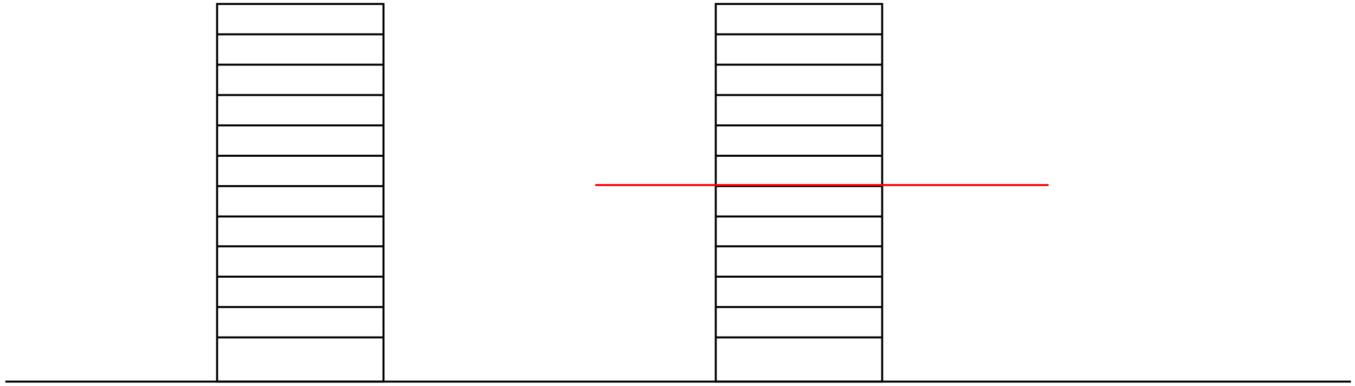
Sensor network, telemetry, GPS, …

Measuring signal amplitude in white noise

Goal of today

Evaluating the theoretical limit of measurement error

Quiz



A (100 coins)

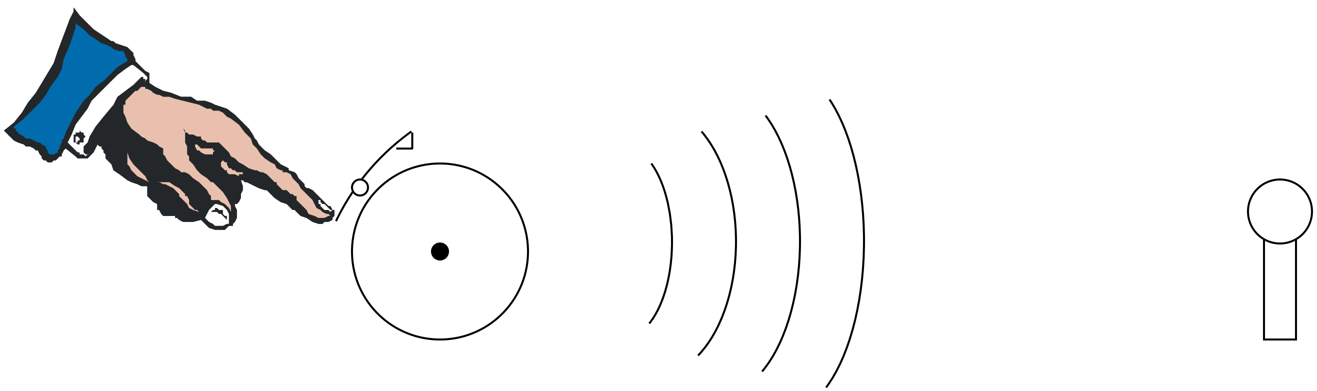
Mixture of
face-up: 30
Face-down: 70

B (100 coins)

Top half: face-up
Bottom half: face-down

You can freely swap coins between A and B or turn them, but you cannot see the coins.
How can you make the numbers of face-up coins and face-down coins equal in both A and B?

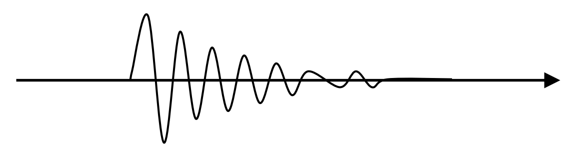
A classical technique called “Synchronous averaging”



- $p(t) = s\,(t) + w(t)$ is observed

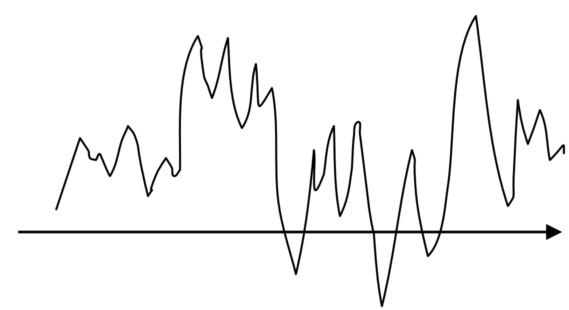
- Finite signal length
- Same waveform every time
- Synchronous signal is obtained

$s\,(t)$



Signal

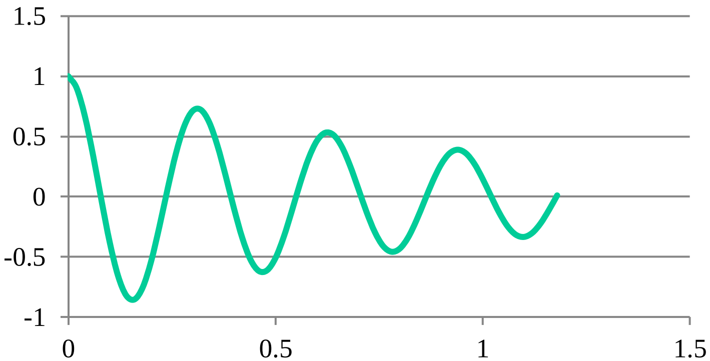
$w\,(t)$

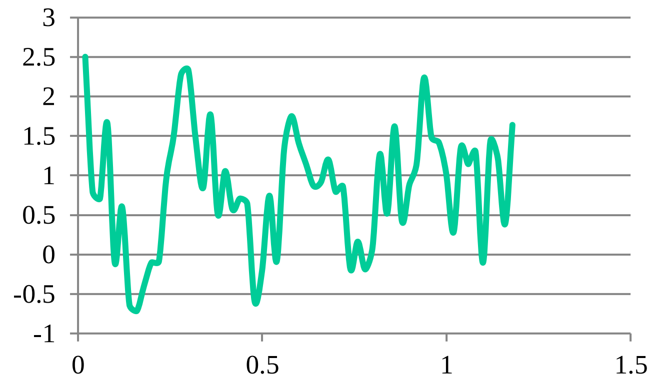


Noise

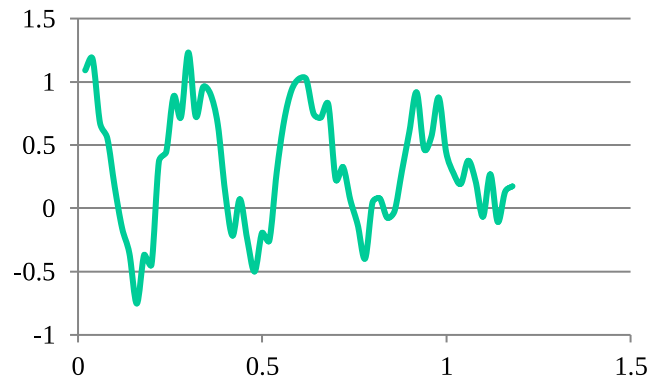
Example of synchronous averaging

Signal

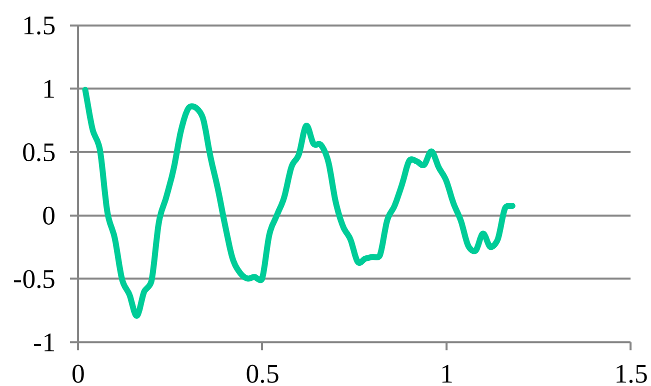




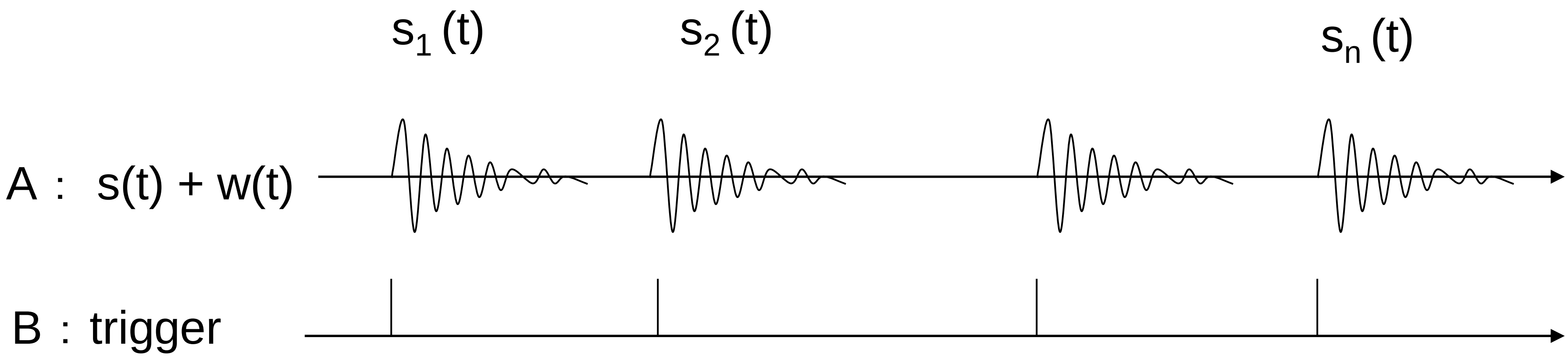
Signal + noise (original)



10 averaging



100 averaging



$$Avg(t) = \frac{1}{N}\sum_{n=1}^{N}\{s_n(t) + w_n(t)\}$$

$$\rightarrow s(t) \quad (N \rightarrow \infty)$$

Synchronous averaging: Averaging values at each time for time-shifted signals so that the trigger signals overlap

Examples of synchronous averaging

Digital oscilloscope

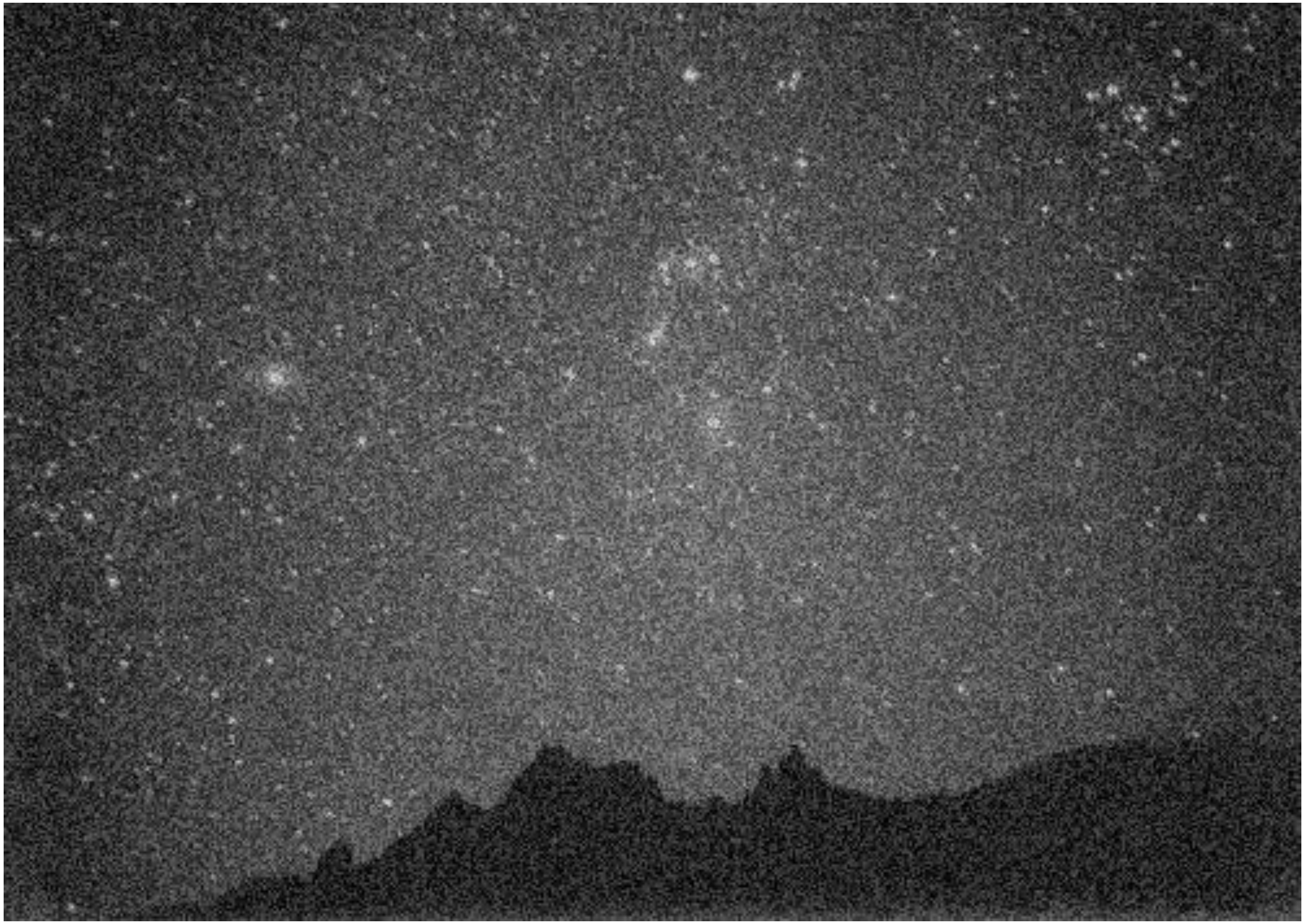
Biological signal measurement

EEG, MEG, Electrocardiography, EMG, …

Non destructive inspection

⋮

Image with noise



After averaging 100 images

Measurement model 1

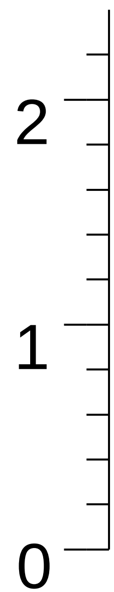
Consider a measured value p is given as

$$p = s + w$$

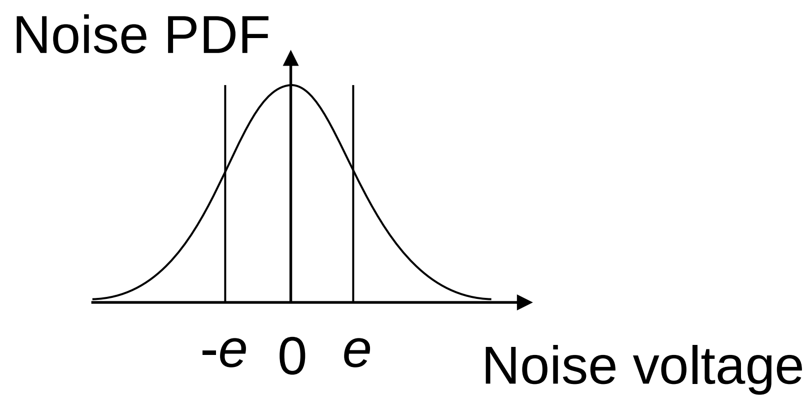
(s: the true value, w: noise).

Measured value *p*

Range: 0 ∼ V_0 [V]



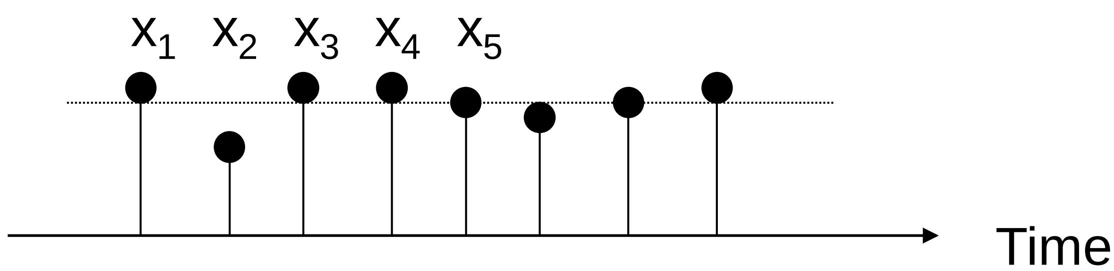
An example of noise PDF
(probability density function)



Central limit theorem for Measurement

If w is a random value, averaging multiple measured values improves the measurement error.

- ○ prior information: True value is constant
- ○ Measure n times
- ○ Noise is random at every measurement



$$e_n = \frac{e_1}{\sqrt{n}}$$

e_n : Error (SD) of n-averaged data, $e_n = \sqrt{E[(p_{\mathrm{AVG}} - s)^2]}$

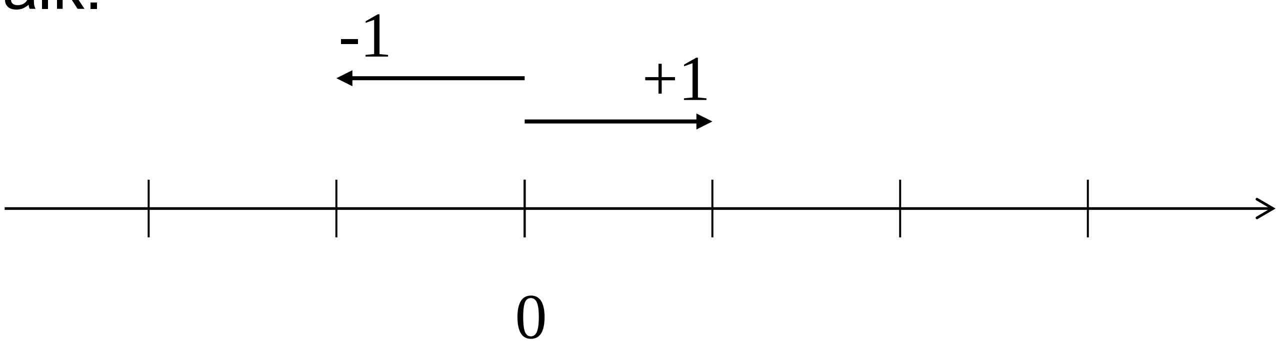
p_{AVG} : Average of n-data

Understanding $e_n = \frac{e_1}{\sqrt{n}}$

e_n : Error (Standard deviation) of n-averaged data

A special case: Random walk

In the case that $w = w_0$ or $-w_0$ (with 50% probability), it is "Random walk."



$d(n)$: Position after n trials $d(n) = a_1 + a_2 + \cdots + a_n, \quad a_i = \pm 1$

Expected value of $d(n)$: $E[d(n)] = 0$

Variance of $d(n)$:

$$V[d(n)] = E[\{d(n)\}^2] = E[a_1^2 + a_2^2 + \cdots + a_n^2] = n$$

General case

Assume w_i: random variable of variance σ^2 and average 0

The average and variance of

$$w = w_1 + w_2 + \cdots + w_n$$

are respectively written as

$$E[w] = 0$$

and

$$V[w] = E[w^2] = E[(w_1 + w_2 + \cdots + w_n)^2]$$

$$= E[w_1^2 + w_2^2 + \cdots + w_n^2] \quad \left(E\left[w_i w_j\right] = 0 \text{ for } i \neq j\right)$$

$$= n\sigma^2.$$

Therefore the standard deviation of $w \propto \sqrt{n}$.

Summary up to now

Assume the observed value is the sum of a constant true value and a random noise. Then the standard deviation of the error of the average of n-measurements decreases in proportion to $\frac{1}{\sqrt{n}}$.

Note: This does not hold unless the noise has a random value in each measurement.

Topic from now

From the case “that the true value is constant”
to the case that “the true value has a certain pattern”

Caution

The pattern ϕ in the next topic might be seen as the signal $s(t)$ shown in “Synchronous averaging,” but that is not so.

The synchronous averaging is an example of the case that the true value is constant.

Measurement model 2

Consider measured values are given as

$$\boldsymbol{p} = \boldsymbol{s} + \boldsymbol{w}$$

that is a N dimensional vector.

$\boldsymbol{p}$: N dimension vector with components of N measure values
$\boldsymbol{s}$: True value (signal)
$\boldsymbol{w}$: White noise independent from $\boldsymbol{s}$

Signal energy $$S = ||\boldsymbol{s}||^2 = \sum_{i=0}^{N-1} s_i^2$$

Noise energy $$W = ||\boldsymbol{w}||^2 = \sum_{i=0}^{N-1} w_i^2$$

Problem 1

Signal $\boldsymbol{s} = A\boldsymbol{\phi}$ is a N dimensional vector composed of N data where $\left\|\boldsymbol{\phi}\right\| = 1$ and A: constant scalar.

Assume $\boldsymbol{\phi}$ is known. Then obtain the minimum value of the signal energy detectable in a white noise whose energy is W, where $\boldsymbol{w}$ is not correlated with $\boldsymbol{s}$.

Answer

It is always possible to decompose the noise $\boldsymbol{w}$ as

$$\boldsymbol{w} = a\boldsymbol{\phi} + \boldsymbol{w}'$$

where $\boldsymbol{w}'$ is a vector orthogonal to $\boldsymbol{\phi}$.

In evaluation of A from the measured N data, the above a is the inevitable error.

Since $\boldsymbol{s}$ and $\boldsymbol{w}$ are independent from each other, the expected value of a^2 is given as

$$E[a^2] = \frac{W}{N}.$$

Therefore, the signal energy S must be larger than W/N to be detected in the white noise with energy W.

Summary up to here

The expected value of noise energy allocated to a component along the signal is give as

$$E[a^2] = \frac{W}{N}$$

- N data signal is considered.
- Random noise is assumed.
- In normal distribution, the probability for

  $$|w_i| > 2.58\sqrt{W/N}$$

  is 1 %.

（Supplementary explanation）

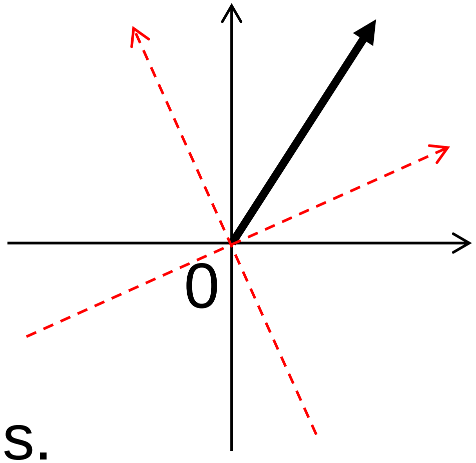
Measured vector $\boldsymbol{p}$ can always be decomposed as

$$\boldsymbol{p} = p_0\boldsymbol{\phi} + p_1\boldsymbol{\varphi}_1 + p_2\boldsymbol{\varphi}_2 + \cdots + p_{N-1}\,\boldsymbol{\varphi}_{N-1}$$

Parallel to the signal

Arbitrarily selected orthonormal basis (orthogonal to $\boldsymbol{\phi}$)

$$\boldsymbol{f} = a_1\boldsymbol{\varphi}_1 + a_2\boldsymbol{\varphi}_2 + \cdots + a_N\,\boldsymbol{\varphi}_N$$



The energy of the signal $\boldsymbol{f}$ is given as

$$F = a_1^2 + a_2^2 + \cdots + a_N^2$$

that is independent on the choice of basis.

（Parseval‘s theorem）

(Supplementary explanation 2)

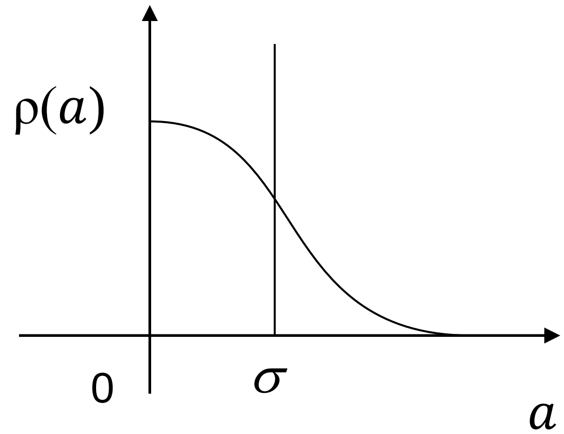
$\rho(a)$: Probability density of a

Parameter a of the former slide is given as

$$a = \boldsymbol{w} \cdot \boldsymbol{\phi} = \sum_{i=0}^{N-1} w_i \phi_i \,.$$

If w_i and w_j $(i \neq j)$ is independent from each other and their variance is σ^2, $\rho(a)$ converges to a Gaussian distribution with variance σ^2 as N increases. (Central limit theorem)

$$\rho(a) \rightarrow \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{a^2}{2\sigma^2}\right\}$$



∗ Note $||\boldsymbol{\phi}|| = 1$.

Derivation of the variance of a in the previous slide

$$E[a^2] = E[(w_0\phi_0 + w_1\phi_1 + \cdots + w_{N-1}\phi_{N-1})^2]$$

$$= E[(w_0\phi_0)^2 + (w_1\phi_1)^2 + \cdots + (w_{N-1}\phi_{N-1})^2]$$

$$\left(\because\ E\left[w_i w_j\right] = 0 \text{ for } i \neq j\right)$$

$$= \sigma^2(\phi_1^2 + \phi_2^2 + \cdots + \phi_n^2)$$

$$= \sigma^2$$

∗ We will change the expressions of $\boldsymbol{s}$ and $\boldsymbol{\phi}$ in the following.

Problem 1-a　Signal amplitude measurement

We measure the effective value of a time-series signal under a white noise with energy W. Obtain the inevitable measurement error of the effective value.

As the prior information, the signal waveform is given as $s(n) = A\phi(n)$ with an unknown parameter A where $n = 0, 2, 3, \ldots, N-1)$

∗ In the following, N-point time-series signal $s(n)$ and $\phi(n)$ are dealt as “N dimensional vectors.”

Answer 1-a

The best estimation of A is given as

$$\overline{A} = \sum_{n=0}^{N-1} p(n)\,\phi(n)$$

with the inevitable error a whose standard deviation is

$$\sigma_A = \sqrt{E[a^2]} = \sqrt{\frac{W}{N}}\,.$$

Therefore, the standard deviation of the estimated effective value (= root mean square value, RMS) is written as

$$\sigma = \frac{\sigma_A}{\sqrt{N}} = \frac{\sqrt{\mathrm{W}}}{N}$$

Question 1

Averaging $f(n)$ for $n = 1, 2, \dots, N$ is equivalent to obtaining the vector component parallel to

$$\phi(n) = C$$

when we consider $f(n)$ and $\phi(n)$ as N dimensional vectors. Confirm this.

Question 2

Consider estimating the signal amplitude from $p(n)$.
Prove that obtaining A that minimizes

$$E = \frac{1}{N}\sum_{n=0}^{N-1}\left\{p(n) - A\phi(n)\right\}^2$$

is equivalent to conducting production of $p(n)$ to $\phi(n)$.

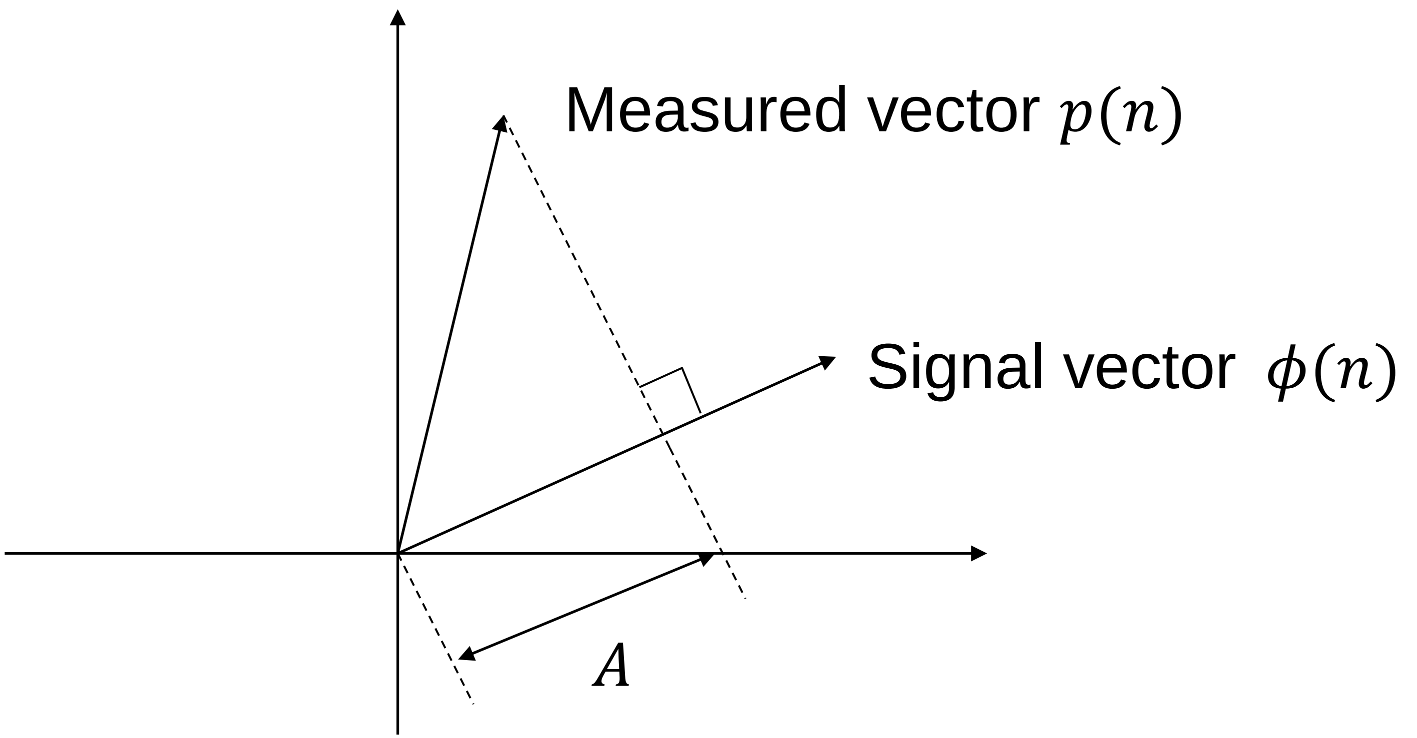
Projection

$$\sum_{n=0}^{N-1} \{p(n) - A\phi(n)\}^2$$

Obtain A to minimize the above

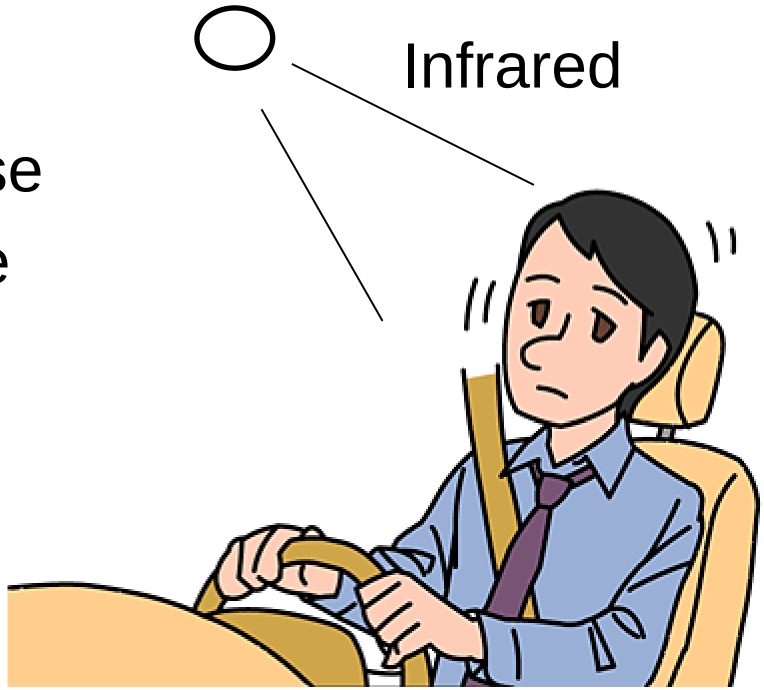
=

Obtain the component included in $f(n)$ parallel to $\phi(n)$.



Application in infrared measurement

Propose a method to decrease the measurement error by the light from the environment.



Problem 2

Consider measuring the phase of a sinusoidal signal at frequency f with energy S. Assume a white noise with energy W is added to the signal. Express the inevitable estimation error of the phase.

Answer

Express the sampled signal (true value) as

$$s(n) = A\cos(Bn+\phi)\ . \quad \longleftarrow \quad B = 2\pi f\ /\ F_S$$

Then the difference of $s(n)$ by a small phase shift $\Delta\phi$ is written as

$$\Delta s(n) = A\cos\{Bn+\phi+\Delta\phi\} - A\cos(Bn+\phi)$$

$$\approx -A\Delta\phi\sin(Bn+\phi)\ , \qquad (1)$$

and the energy of the above signal is

$$E(\Delta\phi) = \sum_{n=1}^{N} \Delta s^2(n) \approx \frac{A^2 N}{2}\Delta\phi^2 = S\Delta\phi^2.$$

Suppose the above expected value of the energy of $\Delta s(n)$ is equal to the energy of the component parallel to $\Delta s(n)$ in the white noise as

$$< E > \approx S < \Delta\phi^2 > = \frac{W}{N}$$

Therefore

$$< \Delta\phi^2 > = \frac{W}{SN}$$

（Cramer-Rao bound）

This is the inevitable estimation error of ϕ.

Note that we assumed $\Delta\phi$ was small and (1) was valid.

General principle

When we estimate a parameter β from a signal $s_\beta(n)$ $(n = 1, 2, \ldots, N)$ under a uncorrelated white noise with energy W, we can not specify β within the error $\Delta\beta$ if the following difference-energy

$$E(\Delta\beta) = \sum_{n=1}^{N} \left\{ s_{\beta+\Delta\beta}(n) - s_\beta(n) \right\}^2$$

is smaller than W/N.

Noise density

What does this mean?

$$2.0\,\frac{\mathrm{V}}{\sqrt{\mathrm{Hz}}}$$

Example:

In the above case, the noise effective value for 100 Hz bandwidth is:

$$2.0\times\sqrt{100}=20\ \ [\mathrm{V}]$$

Question

A sinusoidal wave of a known frequency was observed for T $[s]$ under a white noise with the frequency density d $[\mathrm{V}/\sqrt{\mathrm{Hz}}]$.

Obtain the inevitable estimation error in the effective value.

Answer

Suppose N point data are obtained by sampling the observed signal (after low-pass filtering with the cut-off frequency of B [Hz], the Nyquist rate). Then, the noise energy is written as

$$W = d^2 BN. \qquad (N = 2BT)$$

If the signal phase is known, the estimation error of the signal effective value (the standard deviation of the estimated value) is given as

$$\sigma = \frac{\sqrt{W}}{N} \quad \text{(Problem 1-a)}$$

$$= d\sqrt{\frac{B}{N}} = d\sqrt{\frac{B}{2BT}} = \frac{d}{\sqrt{2T}} \quad [\mathrm{V}]$$

Additional question: How does the result change if the phase is unknown (only the frequency is known)?

Question

Consider estimating the sound source direction from acoustic signals observed at two points.
Assume that it is a two-dimensional problem and the sound source is far enough.

The direction is estimated by the temporal difference of the signals observed at microphones 1 and 2.
Show the inevitable estimation error of the noise source direction assuming the following values.

- Observed signal at microphones 1 and 2:
  Sinusoidal wave of amplitude $1\ \mathrm{V}$ and $500\ \mathrm{Hz}$
- Observed signal duration: $0.1\ s$
- Noise at the microhphones: $0.01\ \mathrm{V}/\sqrt{\mathrm{Hz}}$